A Top-Down Approach to Search-Trees

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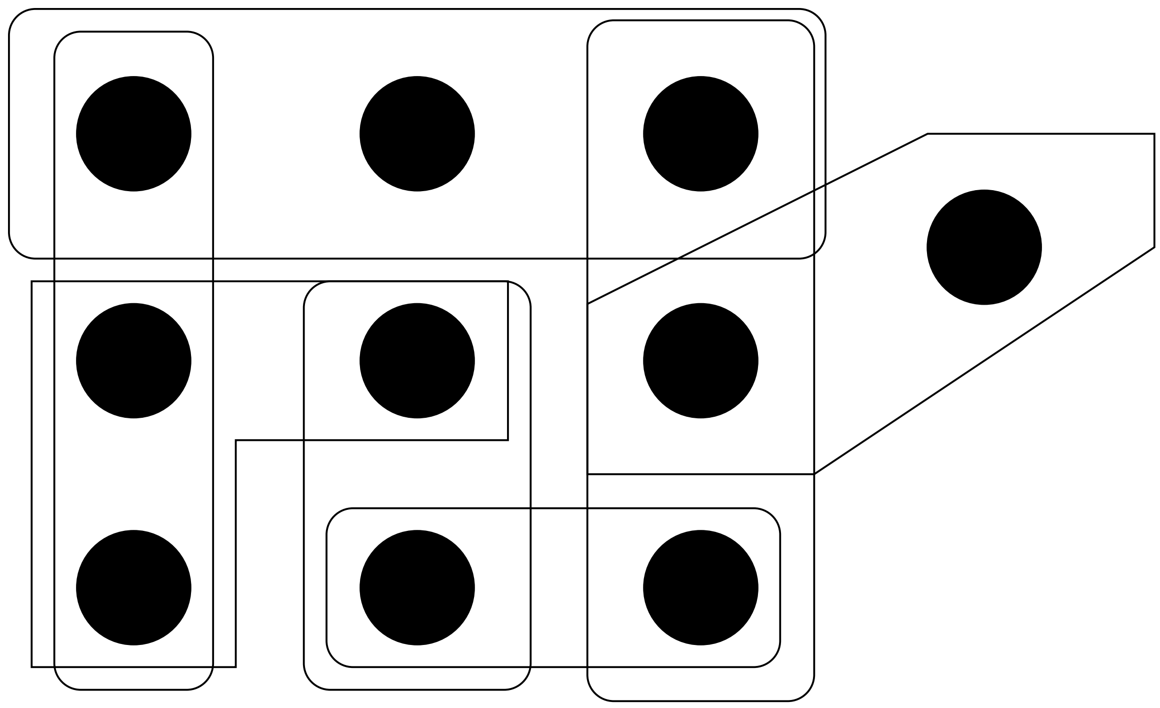
Overview

- HITTING SET: The problem
- Data reduction rules
- Heuristic priorities
- HITTING SET: The algorithm
- Parameterized analysis

d-HITTING SET

Given: A hypergraph $G = (V, E)$ with *edge degree* bounded by d: $\forall e \in E (|e| \leq d)$
Parameter: k
Question: Is there a *hitting set* of size at most k: $\exists C \subseteq V \forall e \in E (C \cap e \neq \emptyset)$?

An example



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Data reduction rules

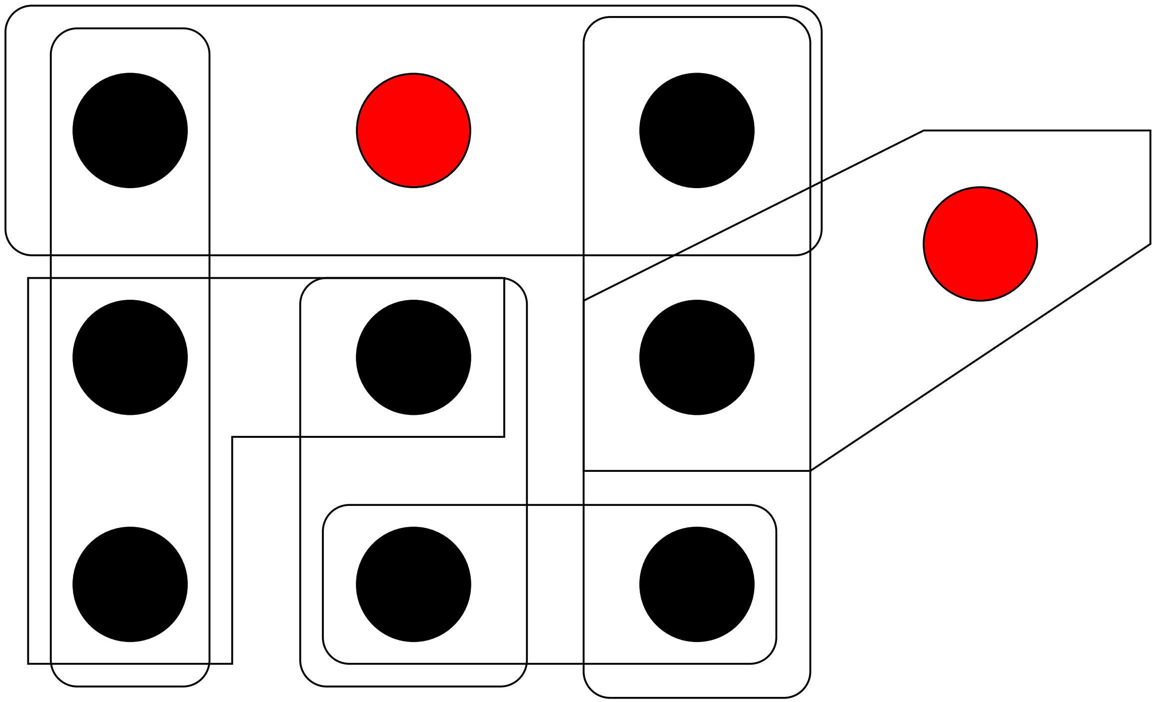
1. (hyper)edge domination: A hyperedge e is *dominated* by another hyperedge f if $f \subset e$. $\leadsto$ delete e

2. small edges: Delete all hyperedges of degree one, place the corresponding vertices into the hitting set.

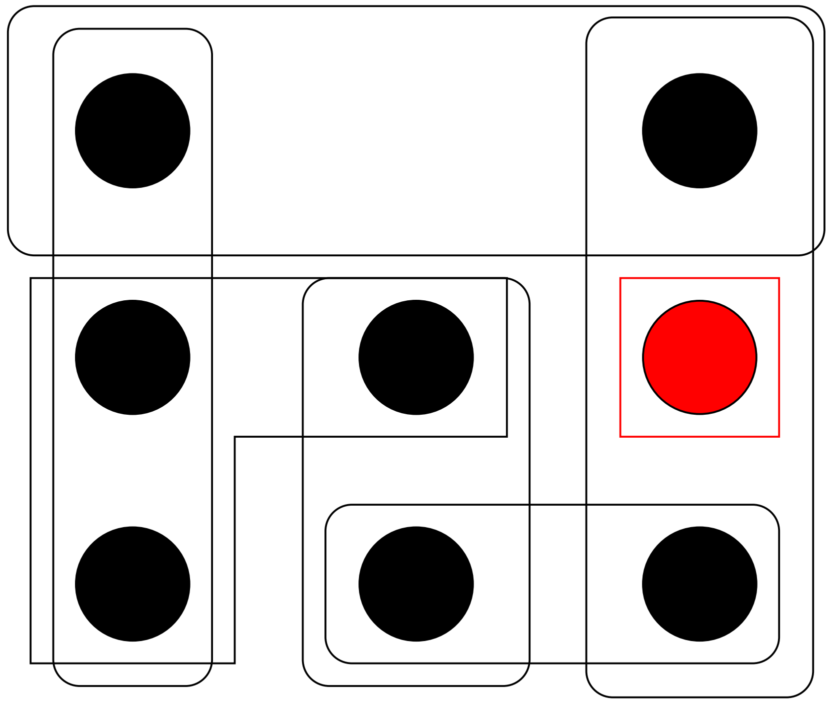
3. vertex domination: A vertex x is *dominated* by a vertex y if, whenever x belongs to some hyperedge e, then y belongs to e, as well. $\leadsto$ delete x

R. S. Garfinkel and G. L. Nemhauser. *Integer Programming*. John Wiley & Sons, 1972.
Often "rediscovered:" K. Weihe (train optimization), R. Niedermeier & P. Rossmanith (parameterized HS, 2003)
Different rules: R. Reiter (HS trees, 1987)

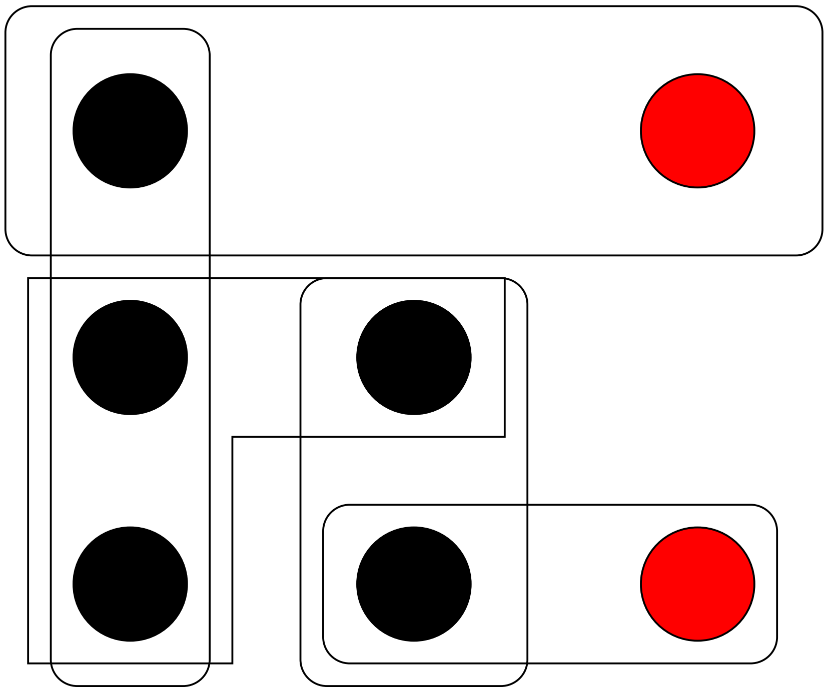
Vertex domination



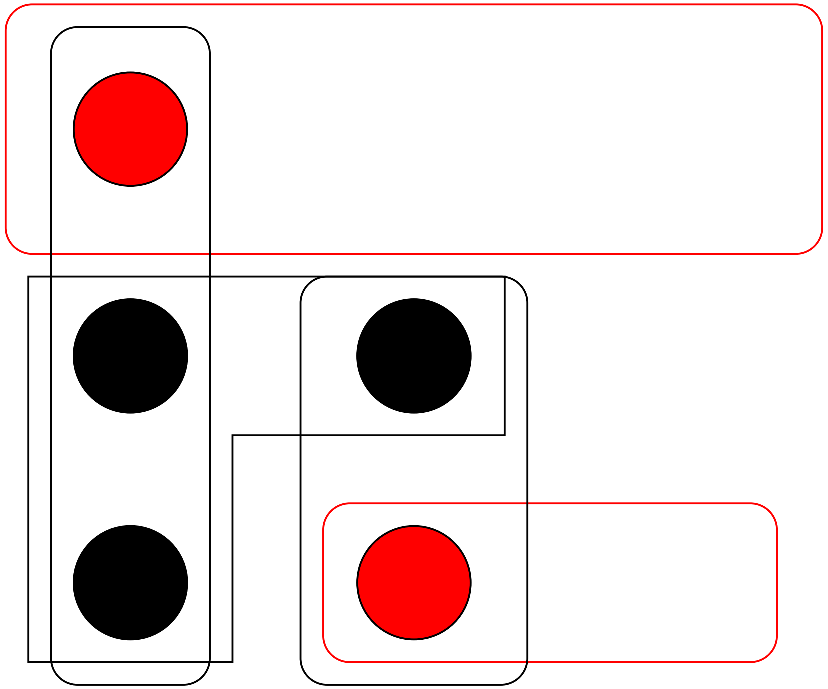
Small edges & edge domination



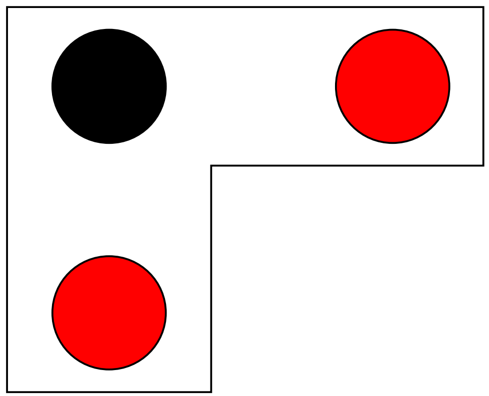
Vertex domination



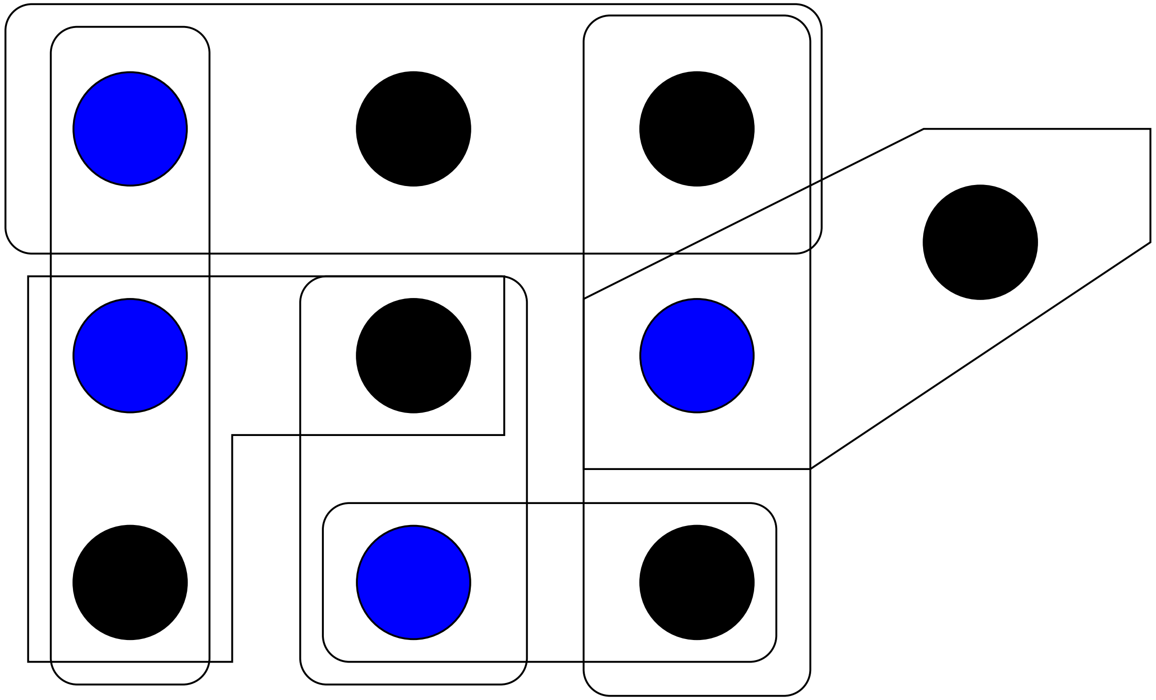
Small edges & edge domination



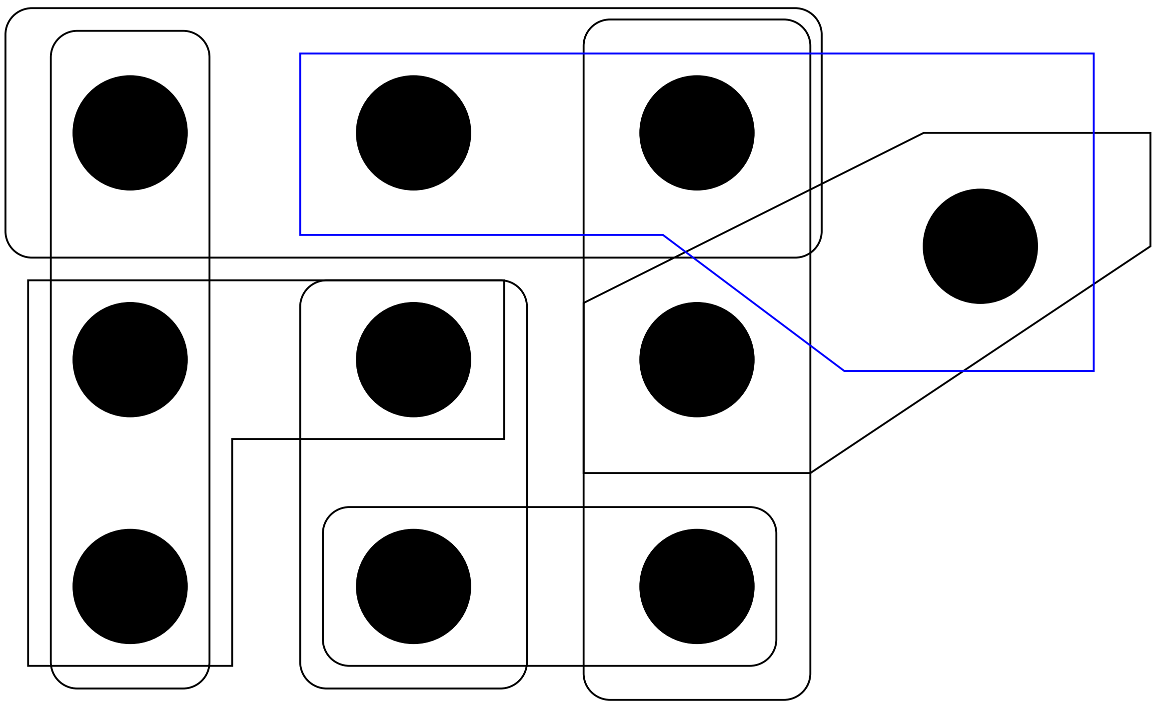
Vertex domination



Reduction solution



Irreducible instance



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Heuristic priority rules

1. Branch on smallest edge

2. Branch on vertex of highest degree

3. Branch on vertex of highest “gain” of small-degree edges

or exchange 2. and 3.

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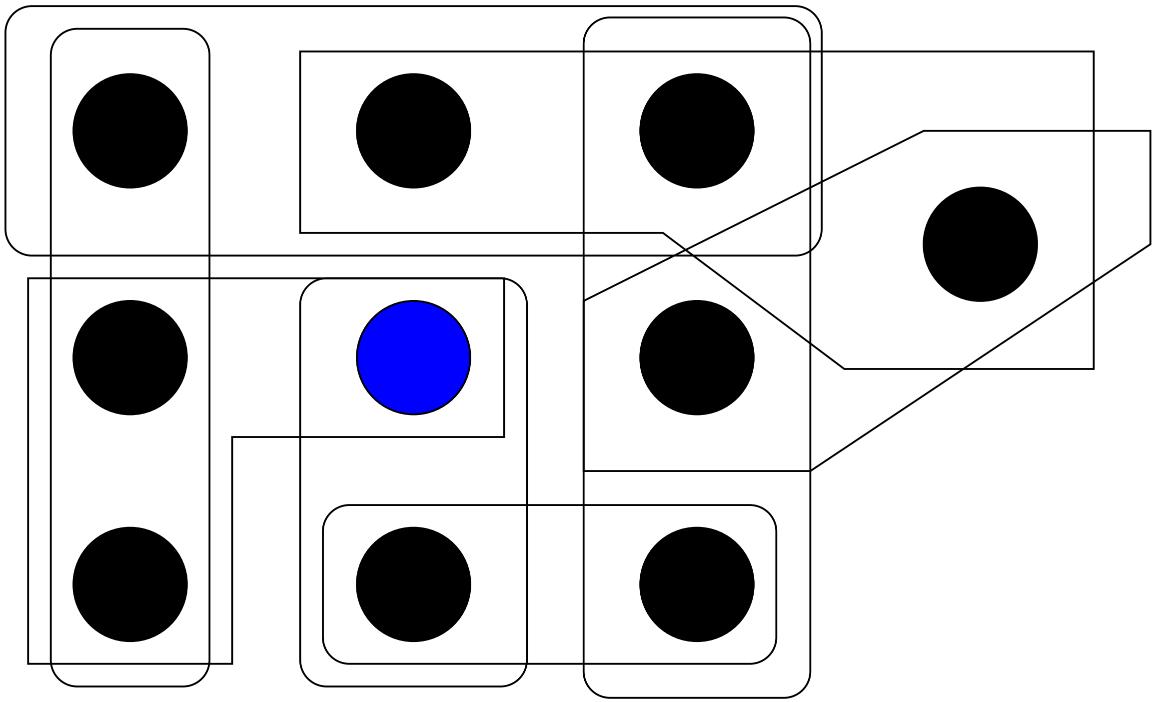
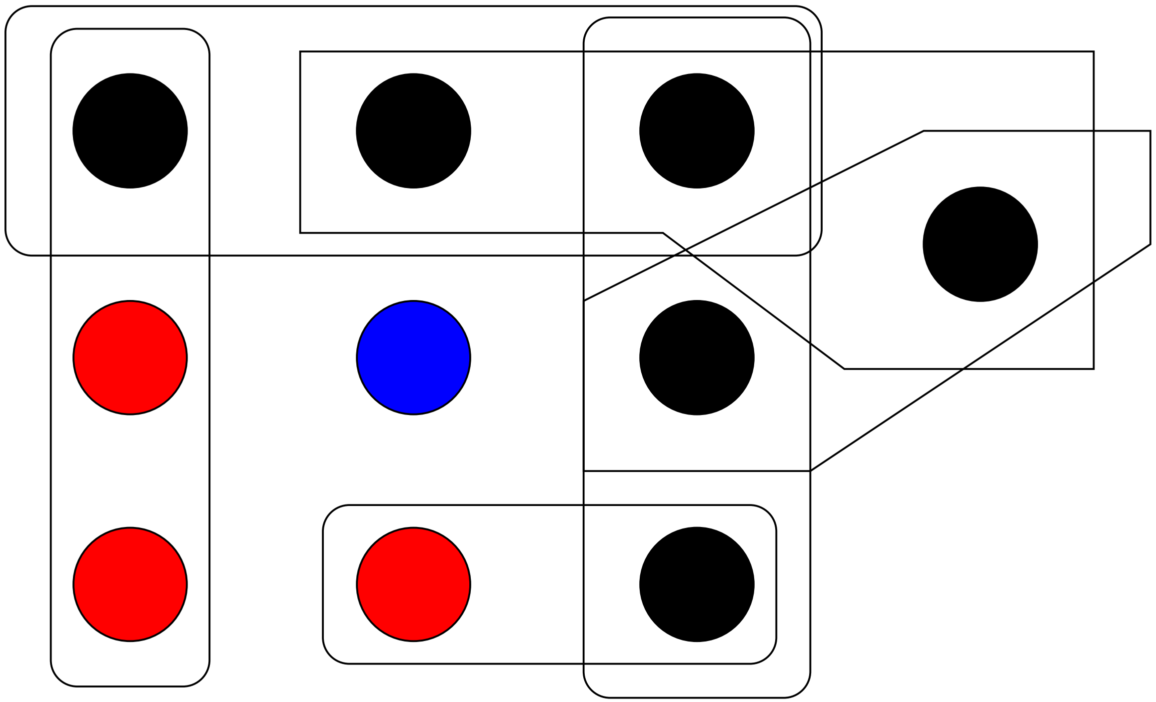
The algorithm HS(G,k)

1. Exhaustively apply all reduction rules, yielding (G',k')
2. **if** $k' \leq 0$ **then** return $(k' = 0 \& E(G') = \emptyset)$
3. Pick $x \in V(G')$ according to heuristic priorities
4. **if** HS$(G' - E[x], k' - 1)$ **then** return YES
5. **else** return HS$(G' - x, k')$

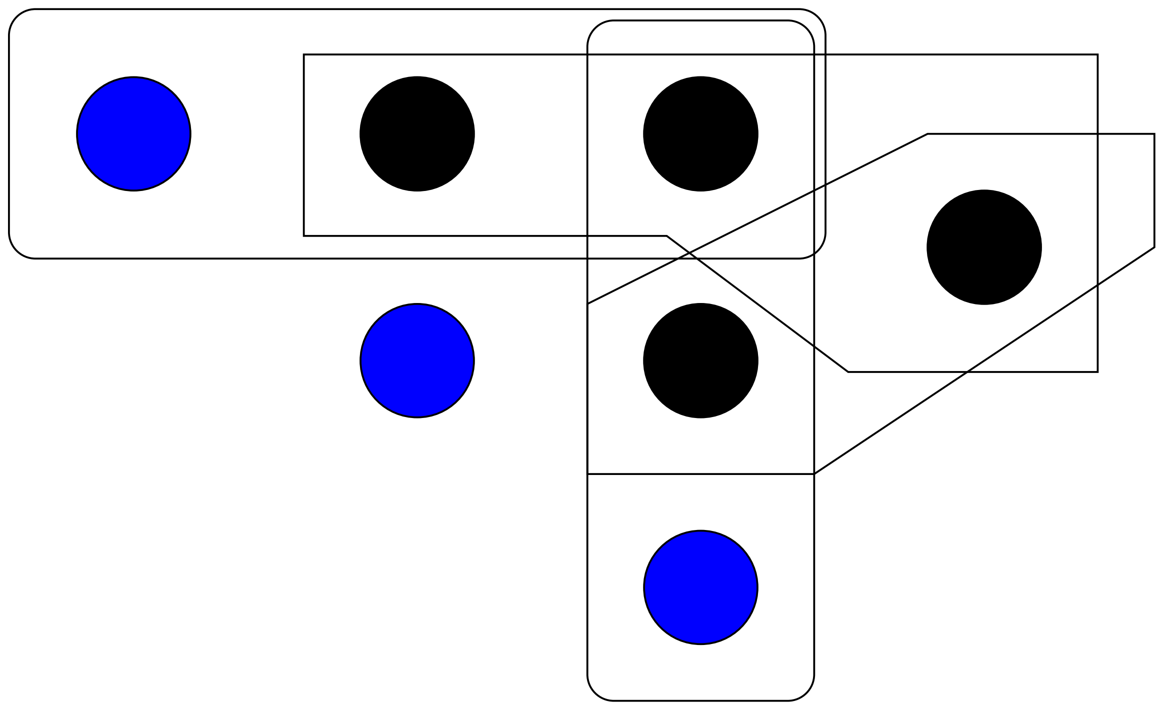
Branching



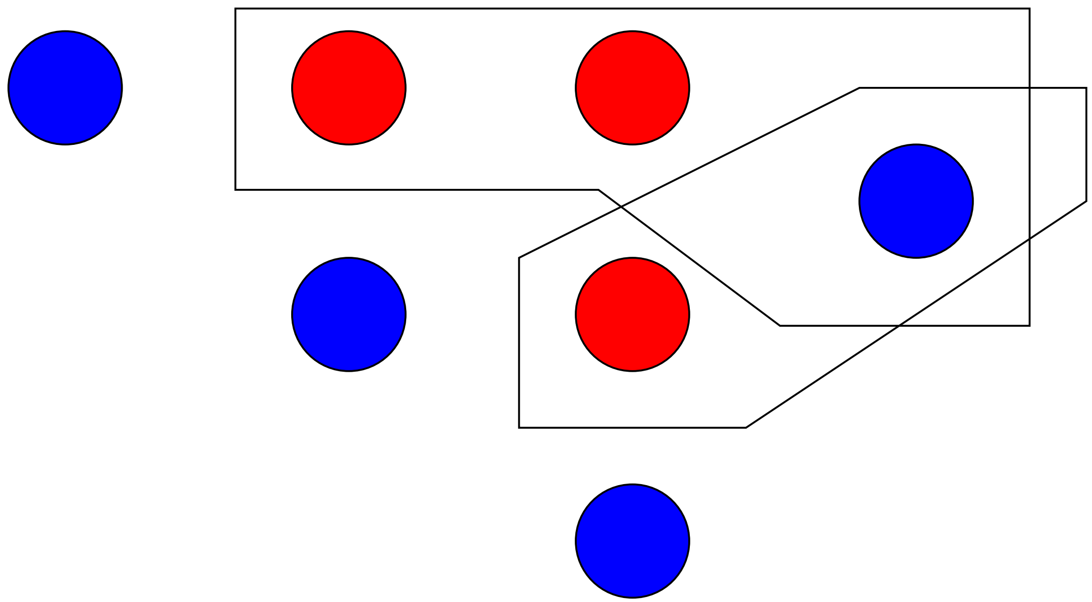
Reduction rules trigger



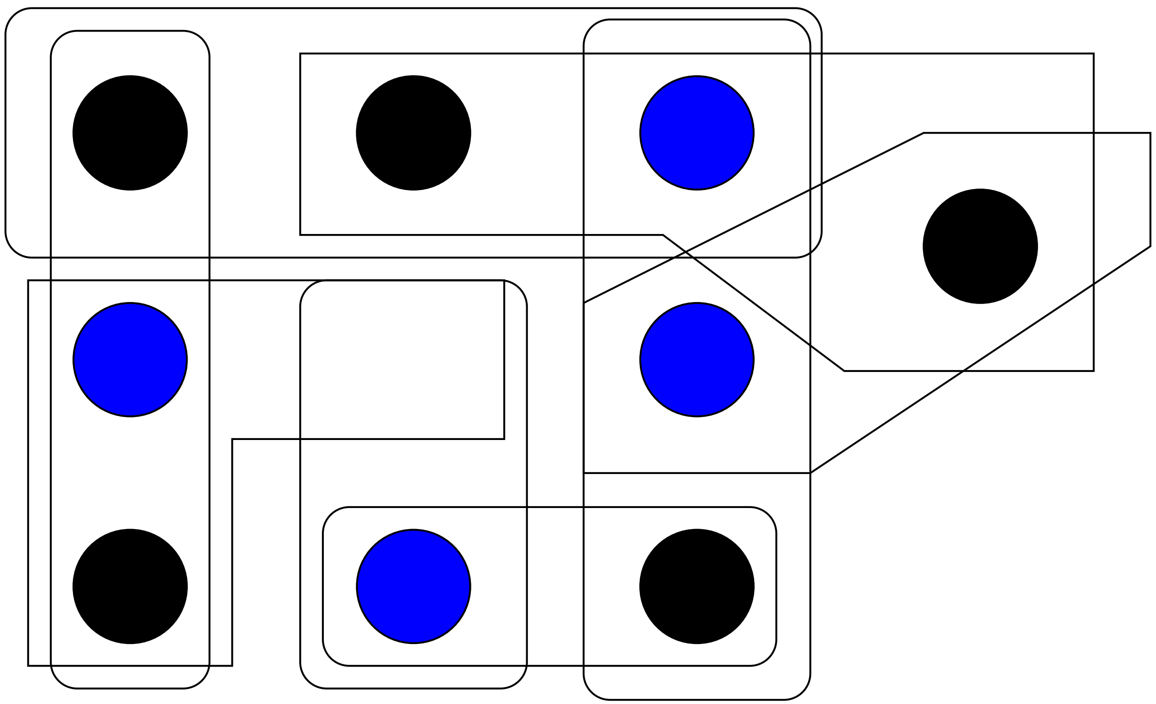
Reduction rules trigger



Reduction rules trigger



A different solution in the other branch



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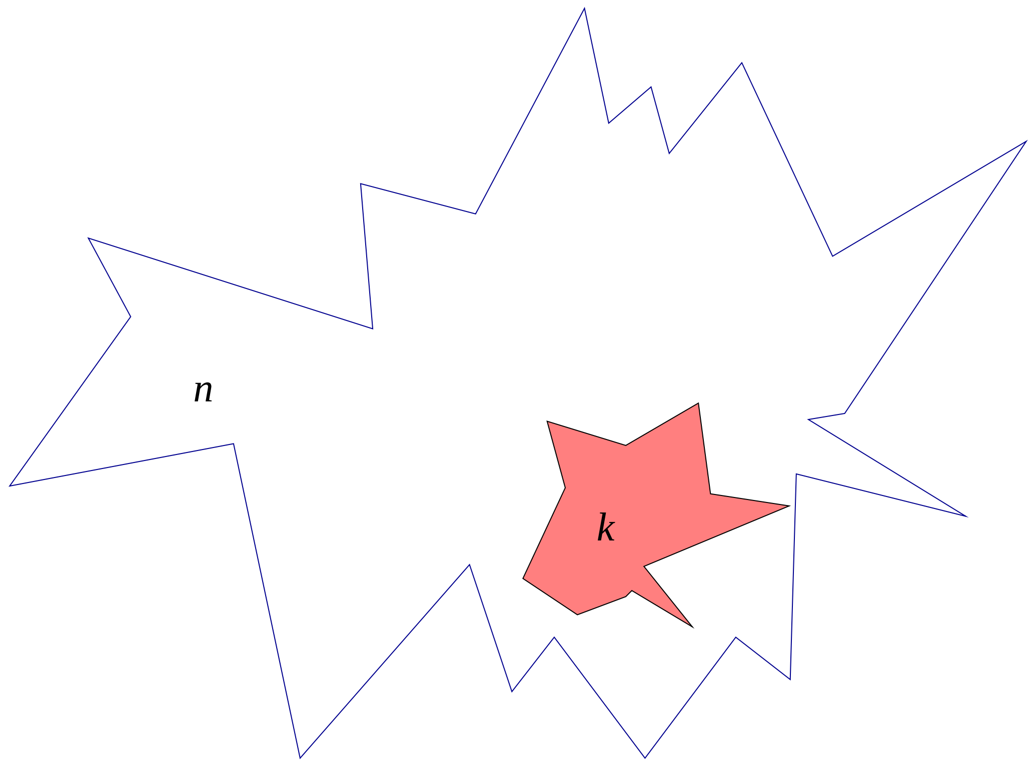
The Curse of Combinatorics



Parameterized complexity in a nutshell

Running time $\mathcal{O}(f(k)p(n))$

Complexity class: FPT

Standard approach: search-trees

Search-tree analysis for d-HITTING SET, parameter k

$T_d^\ell(k)$: # leaves of a search-tree where $\geq \ell$ hyperedges in the given instance (with parameter k) have a degree of (at most) $(d-1)$.

Obvious: $T_d^\ell(k) \leq T_d^{\ell+1}(k)$

We analyze T_d^0, T_d^1 and T_d^2.

T_d^0

$T_d^0 \leq T_d^0(k-1) + T_d^2(k)$.

Branch at x of degree at least two (reduction rules!).

$T_d^0(k-1)$: take x into HS.
$T_d^2(k)$: don't take x into HS, but create two small edges.

T_3^1

$T_3^1 \leq \max\{2T_3^1(k-1), T_3^0(k-1)+T_3^2(k-1)\}$.

Consider edge $e=\{x,y\}$.

case 1: $\delta(x)=\delta(y)=2$.
vertex domination: $x \in$ HS $\iff$ $y \notin$ HS $\leadsto 2T_3^1(k-1)$.

case 2: $\delta(x) \geq 3$, $\delta(y) \leq \delta(x)$: $T_3^0(k-1)+T_3^2(k-1)$

More general: $T_d^1(k) = \max\{(d-1)T^1(k-1), T_d^0(k-1)+(d-2)T_d^2(k-1)\}$.

T_3^2

$$T_3^2(k) \leq \max\{T_3^1(k-1) + T_3^2(k-1), T_3^0(k-2) + T_3^1(k-1), T_3^0(k-1) + T_3^2(k-2)\}$$

case 1: small edges don't intersect

case 2: small edges do intersect; where to branch?

A sample computation for $d = 3$

$$\begin{aligned} T^0(k) &= T^0(k-1) + T^2(k) \\ T^1(k) &= T^0(k-1) + T^2(k-1) \\ T^2(k) &= T^1(k-1) + T^2(k-1) = T^0(k-2) + T^2(k-1) + T^2(k-2) \end{aligned}$$

$\rightsquigarrow (T^0(k) - T^0(k-1)) = (T^0(k-1) - T^0(k-2)) + T^0(k-2) + (T^0(k-2) - T^0(k-3))$
$\rightsquigarrow T^0(k) = 2T^0(k-1) + T^0(k-2) - T^0(k-3)$ which implies $T^0(k) \leq \boxed{2.2470}^k$.

More cases to analyze! T^3-analysis gives better result.

Summary

	d	3	4	5	6	10	100
new	$T_d(k)$	2.206^k	3.12^k	4.06^k	5.04^k	9.01^k	99.0002^k
old	$T_d(k)$	2.270^k	3.31^k	4.24^k	5.20^k	9.11^k	99.0101^k

Further applications I: graph biplanarization

“Nice:” forbidden structure characterization $\rightsquigarrow$ HS

Problem: necessary post-processing of HS solutions

Hence: vertex domination rule is invalid

BUT: still min-degree two for vertices to branch on (application specific lemmas)

New results: 2-sided biplanarization $\mathcal{O}^*(5.050^k)$ old: $\mathcal{O}^*(6^k)$
1-sided biplanarization $\mathcal{O}^*(2.562^k)$ old: $\mathcal{O}^*(3^k)$.

Further applications II: weighted HS

Here: leave vertices of degree one till “the end”

Example: WEIGHTED 3-HITTING SET: $\mathcal{O}^*(2.562^k)$.