

Kleine Kerne und klassische Komplexität

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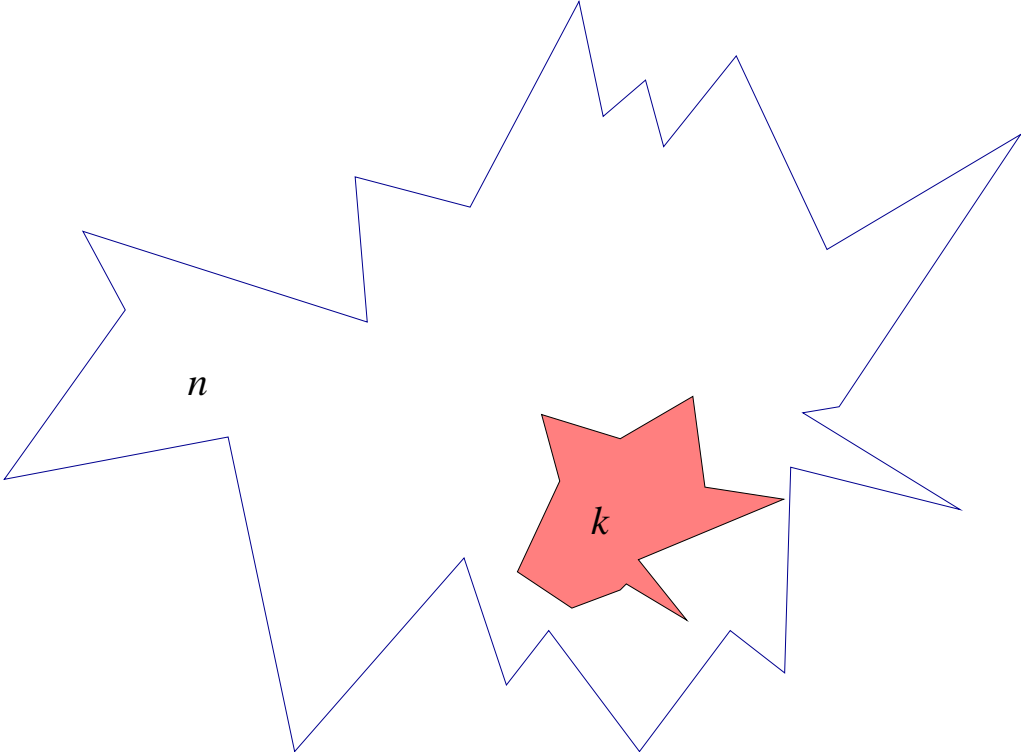
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Overview

- Parameterized complexity in a nutshell
- Duality
- Algorithmics
- Lower bounds on kernel sizes

The Curse of Combinatorics



Parameterized complexity in a nutshell

Running time $\mathcal{O}(f(k)p(n))$

Problem kernel of size $g(k)$, computable in time $q(n)$.

Thm.: Both approaches yield the same.

Complexity class: FPT

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Parametric duality: Intuition

Intuitively clear, but “hard to formalize” notion.

Examples (of vertex selection problems; n is # of vertices):

- k -VERTEX COVER versus
 $(n - k_d)$ -VERTEX COVER = k_d -INDEPENDENT SET
- k -DOMINATING SET versus
 $(n - k_d)$ -DOMINATING SET = k_d -NONBLOCKER

Good news: all problems on planar graphs

Parameterized duality — more formally

If (I, k) is an instance of a parameterized problem P with size function $s(\cdot)$, then (I, k_d) is the corresponding instance of the dual parameterized problem P_d with (dual) parameter $k_d = s(I) - k$.

To make dualization reasonable, we require that $s(I) \geq k$ for all (I, k) .

Observe: keeping the size function, the dual of a dual is the original problem.

We will use the chosen size function also to measure kernel sizes.

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Algorithmics

Thm.: Let P and P_d be a pair of parameterized problems, both in FPT.

f : some monotone function.

A : algorithm for solving (I, k) in time $\mathcal{O}(f(\beta k)p(s(I)))$.

A_d : algorithm for solving (I, k_d) in time $\mathcal{O}(f(\beta_d k_d)p_d(s(I)))$.

\leadsto There is an algorithm A' for solving the non-parameterized problem P in

$$\mathcal{O}\left(f\left(\frac{\beta\beta_d}{\beta + \beta_d}s(I)\right)p'(s(I))\right)$$

time for some polynomial p' .

Algorithm:

For all parameter values k do:

if $k \leq \frac{\beta_d}{\beta + \beta_d} s(I)$ then compute $A(I, k)$

otherwise, compute $A_d(I, s(I) - k)$

output the 'best' of all computed solutions

$$p'(n) = n \max\{p(n), p_d(n)\}$$

Correctness follows by some algebra.

Applications

PLANAR MINIMUM VERTEX COVER can be solved in time $\mathcal{O}(13.68^{\sqrt{n}})$.

Taking the 10^{20} -bound, this means that this algorithm can cope with graphs with over more than 300 vertices.

In comparison, Robson's algorithm would "only" allow up to 250 vertices.

PLANAR MINIMUM DOMINATING SET can be solved in time $\mathcal{O}(5.67^n)$, allowing to solve arbitrary planar graph instances up to about 30 vertices.

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Lower bounds for small kernels

Thm.: Let P (with size function $s(\cdot)$) be \mathcal{NP} -hard.
If each instance (I, k) admits an αk -size kernel,
if its dual (I, k_d) admits an $\alpha_d k_d$ -size kernel, then

$$(\alpha - 1)(\alpha_d - 1) < 1$$

unless \mathcal{P} equals \mathcal{NP} .

Proof of lower bound:

$r(\cdot)$: a pol.-time kernelization for P .

$r_d(\cdot)$: a pol.-time kernelization for P_d .

Consider the following program for a reduction R , given an instance (I, k) of P :

if $k \leq \frac{\alpha_d}{\alpha + \alpha_d} s(I)$ then compute $r(I, k)$
otherwise, compute $r_d(I, s(I) - k)$

For the size of the R -reduced instance I' , we can compute:

- If $k \leq \frac{\alpha_d}{\alpha + \alpha_d} s(I)$, then $s(I') \leq \alpha k \leq \frac{\alpha \alpha_d}{\alpha + \alpha_d} s(I)$.

- Otherwise,

$$\begin{aligned} s(I') &\leq \alpha_d k_d \\ &= \alpha_d (s(I) - k) \\ &< \alpha_d \left(s(I) - \frac{\alpha_d}{\alpha + \alpha_d} s(I) \right) \\ &= \frac{\alpha \alpha_d}{\alpha + \alpha_d} s(I) \end{aligned}$$

By repeatedly applying R , the problem P is solvable in polynomial time,
if $\frac{\alpha \alpha_d}{\alpha + \alpha_d} < 1$.

Lower bounds: applications I

Assuming \mathcal{P} is not equal to \mathcal{NP} , we can conclude:

1. For any $\varepsilon > 0$, there is no $(4/3 - \varepsilon)k$ kernel for PLANAR k -VERTEX COVER.

Proof: PLANAR k_d -INDEPENDENT SET has a $4k_d$ kernel.

2. For any $\varepsilon > 0$, there is no $(2 - \varepsilon)k_d$ kernel for PLANAR k_d -INDEPENDENT SET.

3. non-planar graphs? \rightsquigarrow [algorithmic consequences](#)

4. triangle-free graphs: [best matching bounds](#)

Lower bounds: applications II

Assuming \mathcal{P} is not equal to \mathcal{NP} , we can conclude:

1. There is no $(336/335 - \varepsilon)k_d$ kernel for PLANAR k_d -NONBLOCKER.

Proof: J. Alber, M. Fellows and R. Niedermeier have derived an $335k$ kernel for k -DOMINATING SET ON PLANAR GRAPHS. Hence, the lowerbound follows.

2. There is no $(2 - \varepsilon)k$ kernel for k -DOMINATING SET ON PLANAR GRAPHS.

Proof: C. McCartin has derived a $2k_d$ kernel for k_d -NONBLOCKER on general graphs where the only kernelization rule (how to deal with isolates) does not introduce any possible violations to planarity or degree bounds.

Assumptions of lower bound theorem

Lemma: If P is a parameterized problem with size function $s(\cdot)$ and if P admits a kernelization r such that $s(r(I, k)) \leq \alpha k$ for some $\alpha < 1$, then P is in \mathcal{P} .

Proof: $s(I') \geq k'$ for each instance (I', k') .

\leadsto For $(I', k') = r(I, k)$, $k' \leq s(I') \leq \alpha k < k$.

Conclusions

- Simple algebra gives first lower bound results on kernel sizes.
- Any extensions towards non-linear kernels?
- What about “non-dualizable” FPT problems?
- “(Linear) kernelization schemes”

More ?

see ECCC report